SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(3), p. 119–123.

4021. *Proposed by Arkady Alt.*

Let $(\overline{a}_n)_{n\geq 0}$ be a sequence of Fibonacci vectors defined recursively by $\overline{a}_0 = \overline{a}, \overline{a}_1 = \overline{b}$ and $\overline{a}_{n+1} = \overline{a}_n + \overline{a}_{n-1}$ for all integers $n \geq 1$. Prove that, for all integers $n \geq 1$, the sum of vectors $\overline{a}_0 + \overline{a}_1 + \cdots + \overline{a}_{4n+1}$ equals $k\overline{a}_i$ for some i and constant k.

We received nine correct solutions. We present the solution by David Stone and John Hawkins (joint).

We shall prove that $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1} = L_{2n+1}\bar{a}_{2n+2}$, where L_k is the kth Lucas number. We use some easily proven results. Here, F_k is the kth Fibonacci number.

1. $F_0 + F_1 + \cdots + F_m = F_{m+2} - 1$.
2. $F_{4n+2} = L_{2n+1}F_{2n+1}$
3. $F_{4n+3} = L_{2n+1}F_{2n+2} + 1$
4. $\bar{a}_k = F_{k-1}\bar{a}_0 + F_k\bar{a}_1$ for $k \geq 1$.

Therefore,

$$\begin{aligned}\sum_{k=0}^{m} \bar{a}_k &= \bar{a}_0 + \sum_{k=1}^{m} (F_{k-1}\bar{a}_0 + F_k\bar{a}_1) \\ &= \bar{a}_0 + \left(\sum_{k=1}^{m} F_{k-1}\right)\bar{a}_0 + \left(\sum_{k=1}^{m}\right)\bar{a}_1 \\ &= \bar{a}_0 + (F_{m+1} - 1)\bar{a}_0 + (F_{m+2} - 1)\bar{a}_1 \\ &= F_{m+1}\bar{a}_0 + F_{m+2}\bar{a}_1 - \bar{a}_1 \\ &= \bar{a}_{m+2} - \bar{a}_1\end{aligned}$$

Hence, with $m = 4n + 1$,

$$\begin{aligned}\sum_{k=0}^{4n+1} \bar{a}_k = \bar{a}_{4n+3} - \bar{a}_1 &= F_{4n+2}\bar{a}_0 + F_{4n+3}\bar{a}_1 - \bar{a}_1 \\ &= (L_{2n+1}F_{2n+1})\bar{a}_0 + (L_{2n+1}F_{2n+2})\bar{a}_1 \\ &= L_{2n+1}(F_{2n+1}\bar{a}_0 + F_{2n+2}\bar{a}_1) \\ &= L_{2n+1}\bar{a}_{2n+2}.\end{aligned}$$

Editor's Comments. Various solvers expressed the coefficient of $\bar{a}_{2n+2}$ as L_{2n+1}, $F_{2n} + F_{2n+2}$, and $\frac{F_{4n+2}}{F_{2n+1}}$ and variations of these resulting from different indexing of the Fibonacci sequence. Swylan pointed out that if the word 'constant' is interpreted to mean 'independent of n', then the claim of the problem is false. Perhaps 'scalar' would have been a better word.

4022. *Proposed by Leonard Giugiuc.*

In a triangle ABC, let internal angle bisectors from angles A, B and C intersect the sides BC, CA and AB in points D, E and F and let the incircle of ΔABC touch the sides in M, N, and P, respectively. Show that

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

We received eleven submissions, of which seven were correct, two were incorrect, and two were incomplete. We present the solution by Titu Zvonaru.

Define $x = NA = PA, \quad y = PB = MB,$ and $z = MC = NC$; then

$$BC = y + z, \quad CA = z + x, \quad \text{and} \quad AB = x + y.$$

By the angle bisector theorem we have

$$\frac{FA}{FB} = \frac{CA}{BC} = \frac{z+x}{y+z}, \quad \frac{DB}{DC} = \frac{AB}{CA} = \frac{x+y}{z+x}, \quad \text{and} \quad \frac{EC}{EA} = \frac{BC}{AB} = \frac{y+z}{x+y}.$$

We therefore have to prove that for positive real numbers x, y, z,

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{z+x}{y+z} + \frac{x+y}{z+x} + \frac{y+z}{x+y}\ . \tag{1}$$

After clearing denominators what we must prove reduces to

$$x^2y^4 + y^2z^4 + z^2x^4 + x^3y^3 + y^3z^3 + z^3x^3 \ \geq \ x^3yz^2 + x^2y^3z + xy^2z^3 + 3x^2y^2z^2. \tag{2}$$

By the AM-GM inequality we have

$$\begin{aligned} x^2y^4 + y^2z^4 + z^2x^4 &\geq 3x^2y^2z^2, \\ x^3y^3 + z^3x^3 + z^3x^3 &\geq 3x^3yz^2, \\ y^3z^3 + x^3y^3 + x^3y^3 &\geq 3x^2y^3z, \quad \text{and} \\ z^3x^3 + y^3z^3 + y^3z^3 &\geq 3xy^2z^3\ , \end{aligned}$$

which together imply that (2) holds. Equality holds if and only if $x = y = z$, which immediately implies that the triangle is equilateral.

Editor's Comments. Most submissions reduced our problem to equation (1), but then algebra caused difficulties with two of the faulty arguments. The solution from Salem Malikić neatly avoided calculations by remarking that (1) is known;